Introduction to Compressible Flows

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Introduction to Compressible flows

- **Definition:** A compressible flow is a flow with *variable density*

Compressibility of fluid: $\tau = \frac{1}{\rho}\frac{d\rho}{dp}$

The compressibility is normally defined either at constant temperature or at constant entropy.

Compressibility of water (STP): $5 \cdot 10^{-10} Pa^{-1}$

Compressibility of air (STP) : $1 \cdot 10^{-5} Pa^{-1}$

$$d\rho = \tau \rho dp$$

$$dp = -½\rho d(u^2) \Rightarrow d\rho \approx -\tau\rho^2 d(u^2)$$

Introduction to Compressible flows

- **Compressible flow techniques are important for**
  - Flows near nozzles in gas and steam turbines
  - Flows through valves, intake, exhaust in reciprocating engines
  - Natural gas transmission lines
  - Combustion chambers
  - Aircraft
  - Space shuttle
  - Rockets

Introduction to Compressible flows

- Compressibility is a measure of density change by a change in pressure.
  - Liquids have a low compressibility.
  - Gases are usually compressible.

- Many industrial and daily life flows are subject to <u>small pressure</u> <u>and density change</u> and can thus be considered as being <u>incompressible</u>.

- For compressible flows, temperature has a significant influence on the flow and is an intrinsic part of the analysis.

- Thermodynamics is a prerequisite for all analyses

Basic Thermodynamic Properties

- Ideal gas law:

$$p = \rho R T$$
$$R = \Re / M; \quad \Re = 8312.3 J/(kmol \cdot K)$$

- Specific heats:

$$c_p(T) = \left.\frac{\partial h}{\partial T}\right|_p; \quad c_v = \left.\frac{\partial e}{\partial T}\right|_v$$

- Calorically perfect gas: c_v and c_p are constants

- Ratio of specific heats: $\gamma \equiv c_p / c_v$

$$p = \rho R T \Rightarrow \frac{p}{\rho} = RT; h \equiv e + \frac{p}{\rho} = e + RT \Rightarrow c_p = c_v + R$$

$$c_p = \frac{\gamma R}{\gamma - 1}; \quad c_v = \frac{R}{\gamma - 1}$$

Basic Thermodynamic Properties

- Entropy (reversible process): $\delta Q = T\delta S$

$$dE = dQ + dW = TdS - pdv$$

$$dh = Tds + 1/\rho dp = c_p dT \Rightarrow \boxed{ds = \frac{c_p dT}{T} - \frac{1}{\rho T}dp = \frac{c_p dT}{T} - \frac{R}{p}dp}$$

$$\boxed{s = c_p \ln\left(T / p^{\frac{\gamma-1}{\gamma}}\right) + const}$$

$$\boxed{s_2 - s_1 = c_p \ln\left[\frac{T_2}{T_1}\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

Basic Thermodynamic Properties

- For an isentropic (adiabatic and reversible process) flow the entropy is constant

$$Tds = de - \frac{p}{\rho^2} d\rho = c_v dT - \frac{p}{\rho^2} d\rho = 0$$

$$Tds = dh - \frac{1}{\rho} dp = c_p dT - \frac{1}{\rho} dp = 0$$

$$\left(\frac{\partial p}{\partial \rho} \right)_s = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \Rightarrow \quad \boxed{\frac{p}{\rho^\gamma} = const}$$

Speed of Sound

- Speed of sound: $a^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$
- Alternatively: $a^2 = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T$
- For a calorically perfect gas: $a = \sqrt{\gamma p / \rho} = \sqrt{\gamma R T}$
- Mach number: $M \equiv V / a$
- Speed of sound (STP)

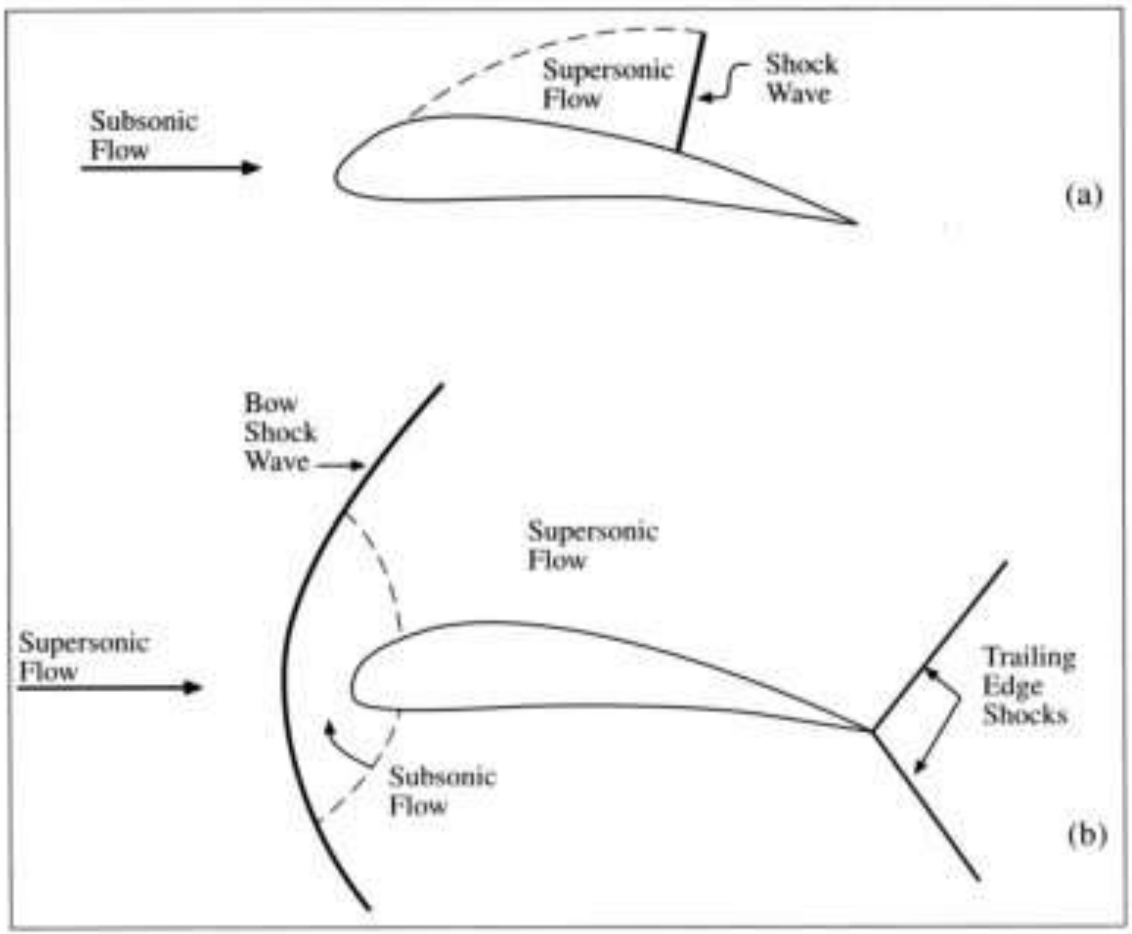
  Water: 1480 m/s

  Air: 340 m/s

Mach number regimes

- Mach number: $M \equiv V / a$
  - M<0.3 incompressible
  - M<1 subsonic
  - M=1 transonic
  - M>1 supersonic
  - M>5 hypersonic

Shock Waves on Airfoil



Acoustic waves in compressible flows

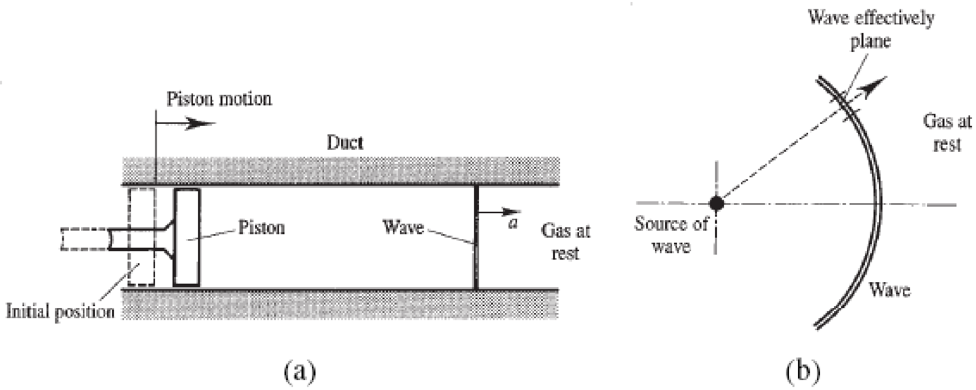


Figure 1: Weak pressure waves generated by (a) the motion of a piston in a duct (plane wave) and (b) by a point source (spherical wave).

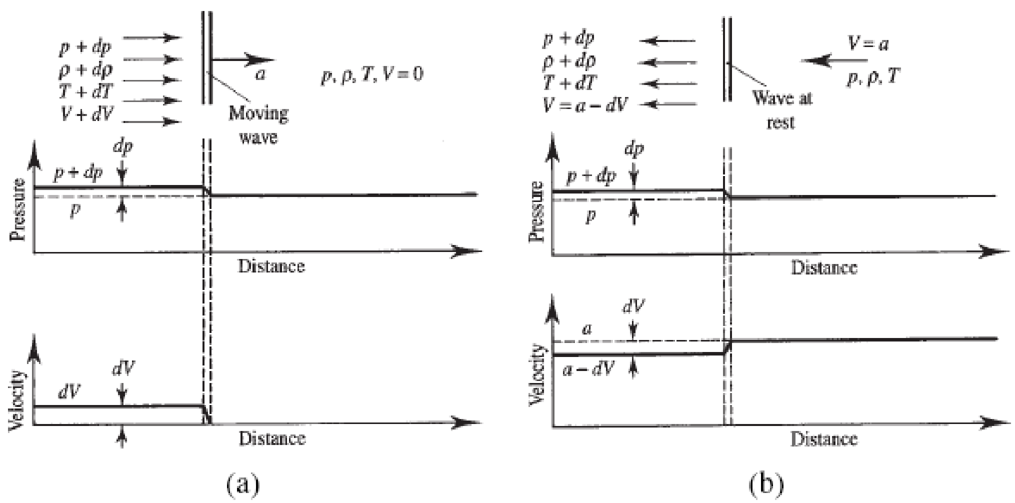


Figure 2: Changes through a propagating weak pressure wave (a) in a fixed coordinate system and (b) relative to the weak pressure wave.

Acoustic waves in compressible flows

- Conservation of mass in the CV:

$$\frac{\dot{m}}{A} = \rho a = (\rho + d\rho)(a - dV) \Rightarrow d\rho = \frac{\rho}{a} dV$$

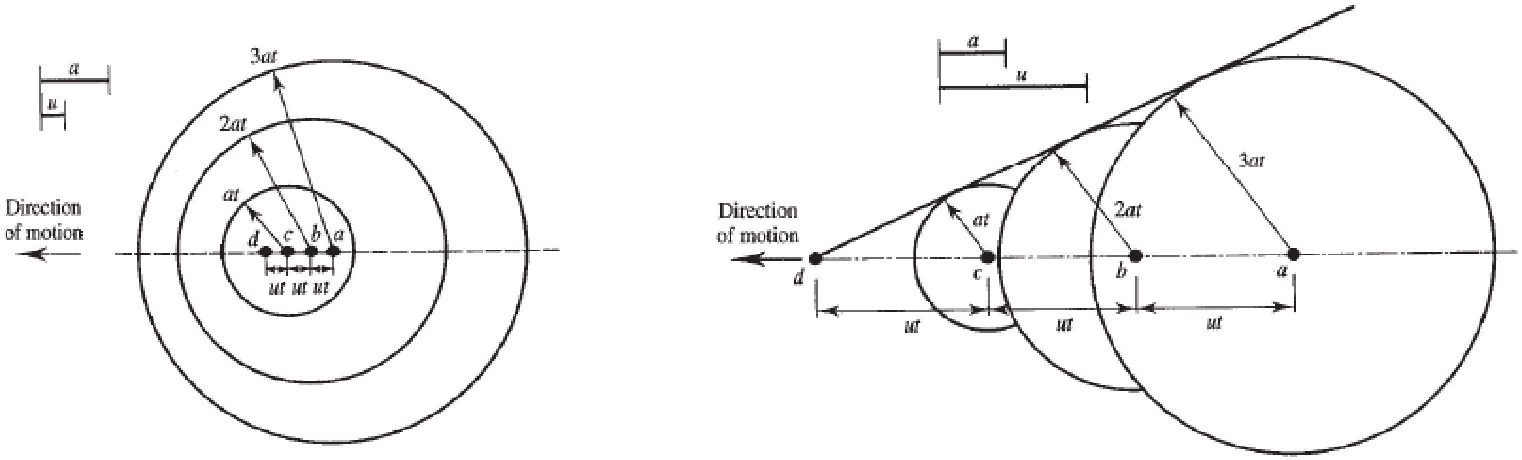
- Conservation of momentum:

$$p - (p + dp) = \left(\frac{\dot{m}}{A}\right)\left[(a - dV) - a\right] \Rightarrow dp = \rho a dV$$

- Combing the two gives

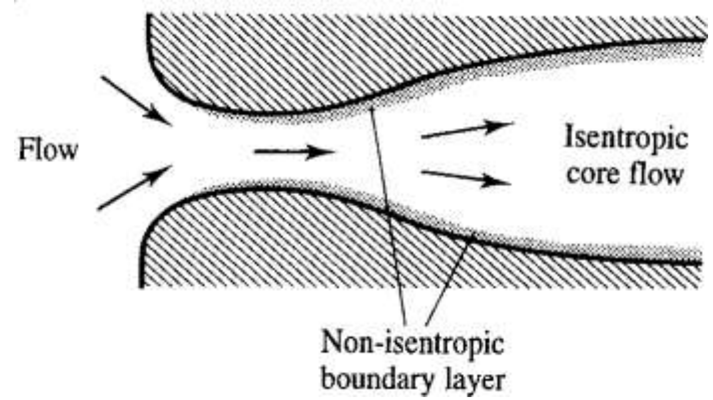
$$\frac{dp}{d\rho} = a^2$$

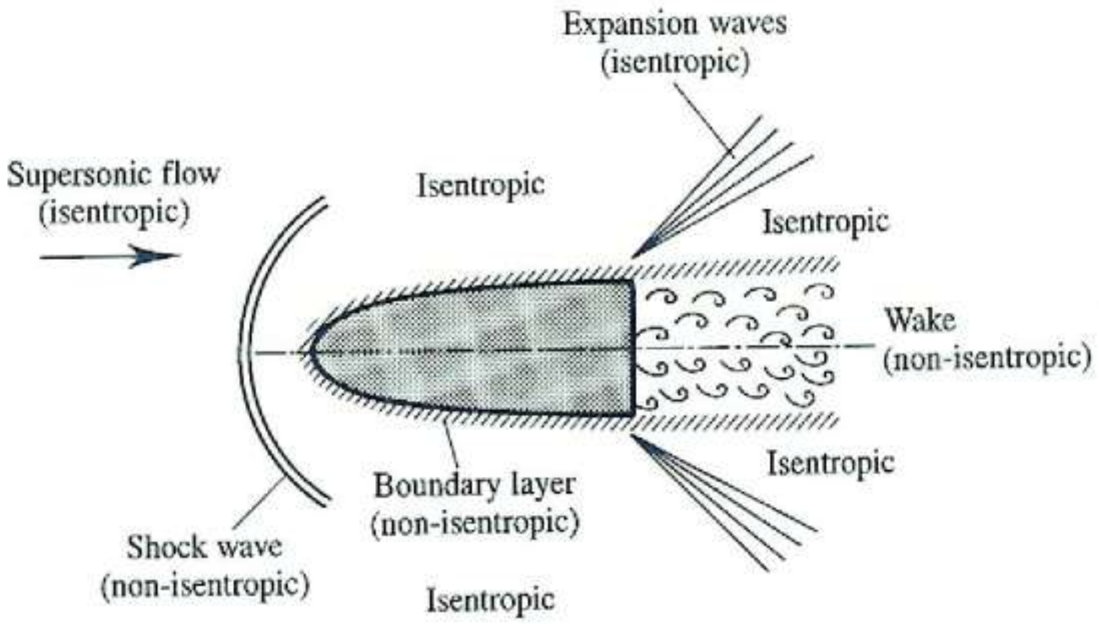
Acoustic waves in compressible flows



- Mach angle: $$\sin\alpha = \frac{a}{V} = \frac{1}{M}$$

1D isentropic compressible flows





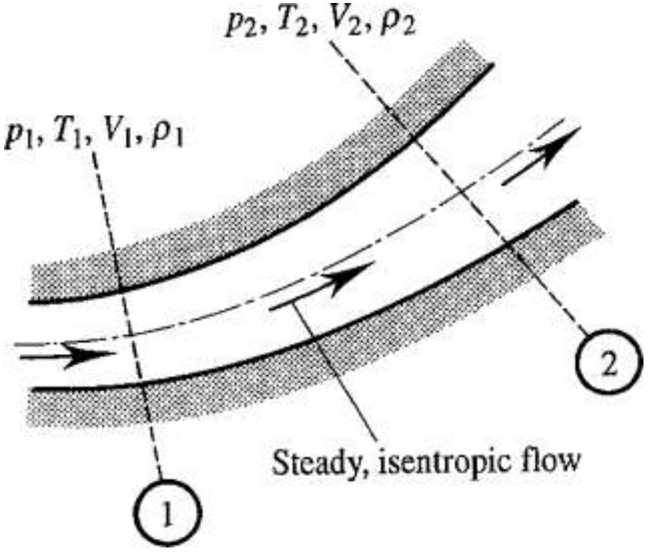
1D isentropic compressible flows

- Isentropic relation (1): $$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$
- Ideal gas law (2):

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \Rightarrow \frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

- (1)+(2) gives

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$



1D isentropic compressible flows

- Energy equation (adiabatic) :

$$\frac{dQ}{dm}+\frac{dW}{dm}=\left(h+\frac{1}{2}V^2+\not{g}\not{z}\right)_2-\left(h+\frac{1}{2}V^2+\not{g}\not{z}\right)_1$$

$$\Rightarrow \quad c_pT_1+\frac{V_1^2}{2}=c_pT_2+\frac{V_2^2}{2}\Rightarrow c_pT_1\left(1+\frac{V_1^2}{2c_pT_1}\right)=c_pT_2\left(1+\frac{V_2^2}{2c_pT_2}\right)$$

- As $\frac{V^2}{2c_pT}=\frac{(\gamma-1)V^2}{2\gamma RT}=\frac{\gamma-1}{2}\frac{V^2}{a^2}=\frac{\gamma-1}{2}M^2$

$$\boxed{\frac{T_2}{T_1}=\frac{1+(\gamma-1)/2\,M_1^2}{1+(\gamma-1)/2\,M_2^2}}$$

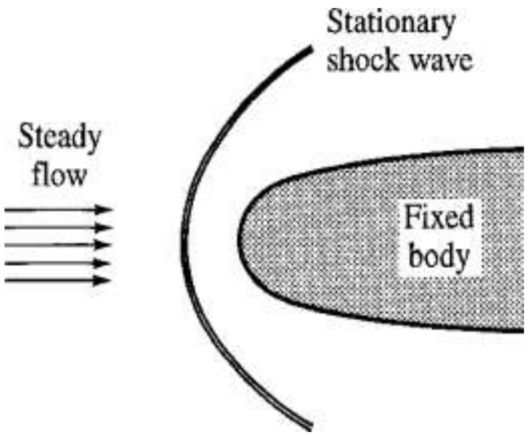
$$\boxed{\frac{p_2}{p_1}=\left(\frac{1+(\gamma-1)/2\,M_1^2}{1+(\gamma-1)/2\,M_2^2}\right)^{\frac{\gamma}{\gamma-1}}} \qquad \boxed{\frac{\rho_2}{\rho_1}=\left(\frac{1+(\gamma-1)/2\,M_1^2}{1+(\gamma-1)/2\,M_2^2}\right)^{\frac{1}{\gamma-1}}}$$

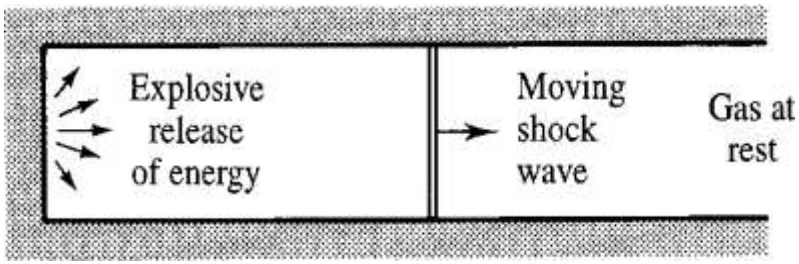
1D isentropic compressible flows

- Continuity equation: $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$

$$\Rightarrow \quad \frac{A_1}{A_2} = \frac{\rho_2 V_2}{\rho_1 V_1} = \frac{M_2}{M_1}\left(\frac{1+(\gamma-1)/2\, M_1^2}{1+(\gamma-1)/2\, M_2^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

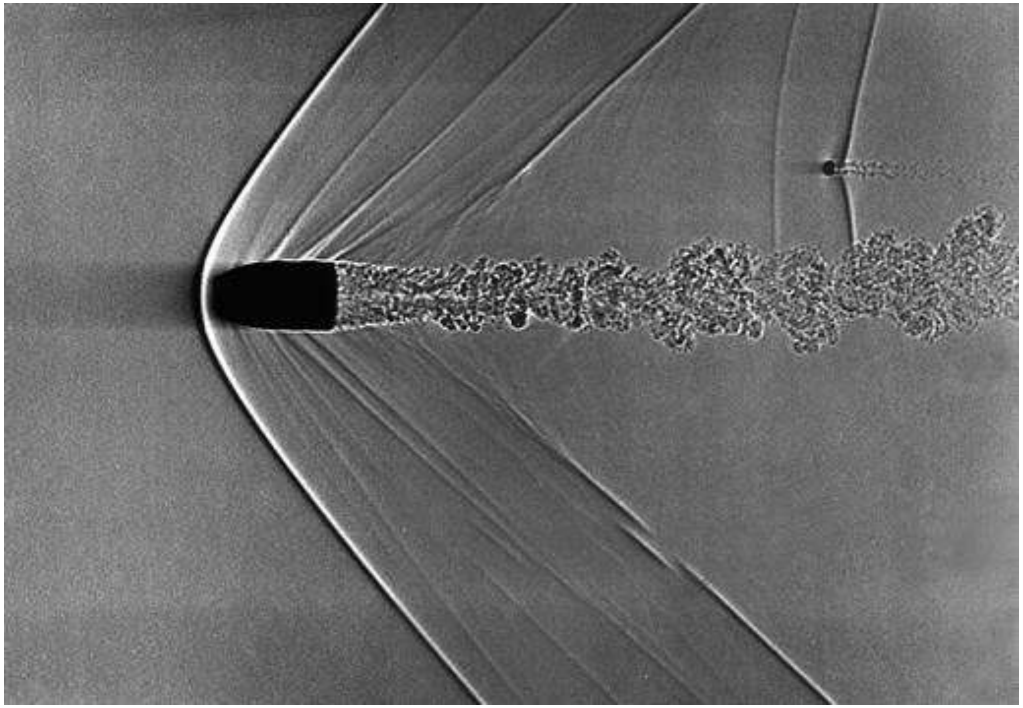
Introduction to shock waves





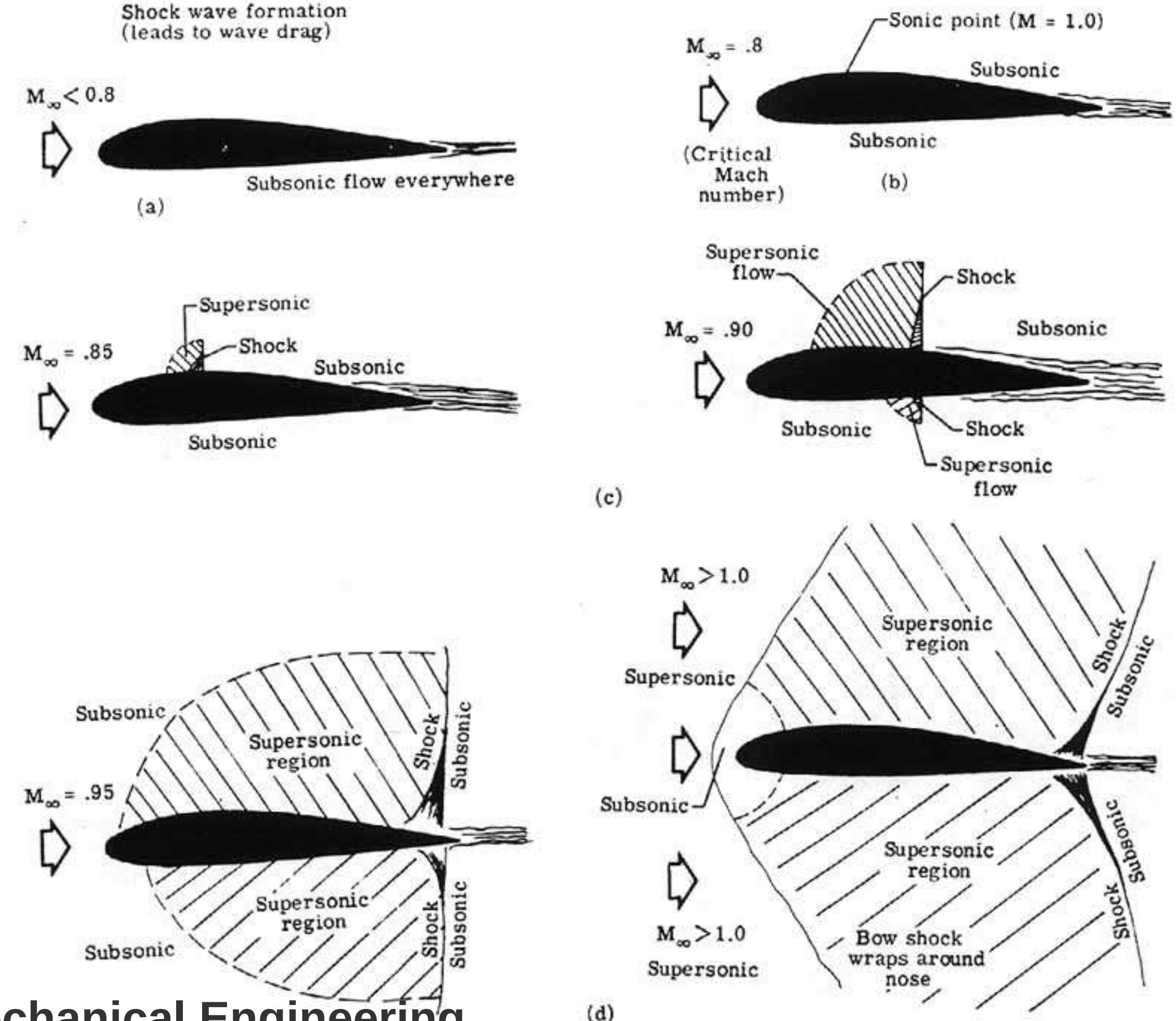
- The high pressure cannot propagate as weak pressure.
- Instead it creates sudden change or discontinuity in pressure and velocity.
- The thickness of a shock wave is about 4 mean free paths (0.2μm), $\Rightarrow$ it is treated as a discontinuity.

Example of compressible flow with shock

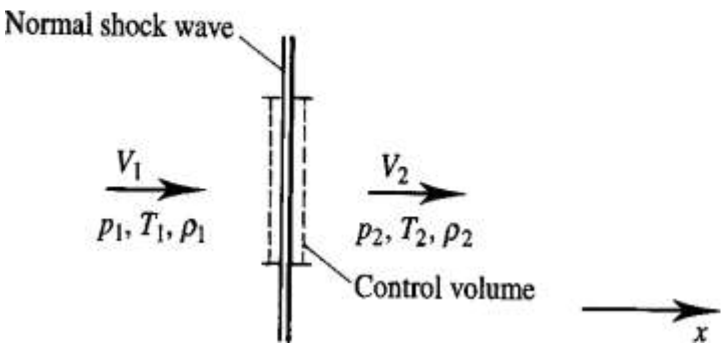


A bullet traveling through air at about 1.5 times the speed of sound

Shock Wave Formation on Airfoil



Normal shock: Basic relations



- Conservation of mass:

$$\frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2$$

- Conservation of momentum:

$$p_1 A - p_2 A = \dot{m}(V_2 - V_1)$$

$$\Rightarrow \quad p_1 - p_2 = \rho_1 V_1 (V_2 - V_1) = \rho_2 V_2 (V_2 - V_1)$$

$$\Rightarrow \quad V_1 V_2 - V_1^2 = \frac{p_1 - p_2}{\rho_1}$$

$$V_2^2 - V_1 V_2 = \frac{p_1 - p_2}{\rho_2}$$

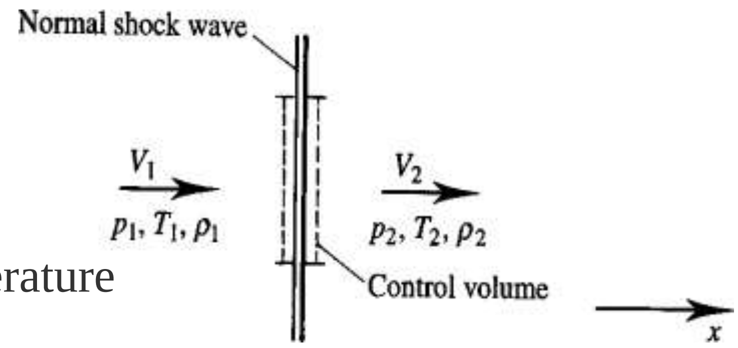
$$\Rightarrow \quad V_2^2 - V_1^2 = \left(p_1 - p_2\right)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Normal shock: Basic relations

- Conservation of energy:

$$c_p T_1 + \frac{V_1^2}{2} = c_p T_2 + \frac{V_2^2}{2} = c_p T_0 = const$$

Stagnation temperature



$$\Rightarrow \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{V_1^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{V_2^2}{2}$$

$$\Rightarrow \boxed{V_2^2 - V_1^2 = \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)}$$

- (Momentum)+(Energy)

$$\Rightarrow \boxed{\frac{\rho_2}{\rho_1} = \frac{\left[\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1\right]}{\left[\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}\right]}}$$

p_2/p_1: strength of the shock wave

Normal shock: Basic relations

- Conservation of mass or continuity:

$$\frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} = \frac{\left[\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1\right]}{\left[\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}\right]}$$

- Perfect gas law:

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{\left[\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}\right]}{\left[\frac{\gamma+1}{\gamma-1}+\frac{p_1}{p_2}\right]}$$

These expressions are referred to as the Rankine-Hugoniot normal shock wave relations.

Normal shock in terms of Mach number

- The ratio of density is:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$$

- The ratio of temperature is:

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_1^2 - (\gamma-1)\right]\left[2+(\gamma-1)M_1^2\right]}{(\gamma+1)^2 M_1^2}$$

- The ratio of sound speed is determined as

$$\frac{a_2}{a_1} = \frac{\sqrt{\left[2\gamma M_1^2 - (\gamma-1)\right]\left[2+(\gamma-1)M_1^2\right]}}{(\gamma+1)M_1} \qquad (**)$$

Normal shock in terms of Mach number

- The downstream Mach number is found from the energy relation.

$$c_p T_2 + \frac{V_2^2}{2} = c_p T_1 + \frac{V_1^2}{2} \Rightarrow \frac{a_2^2}{\gamma - 1} + \frac{V_2^2}{2} = \frac{a_1^2}{\gamma - 1} + \frac{V_1^2}{2}$$

$$\frac{a_2^2}{\gamma - 1} + \frac{V_2^2}{2} = \frac{a_1^2}{\gamma - 1} + \frac{V_1^2}{2} \Rightarrow \frac{a_2^2}{a_1^2}(1 + \frac{\gamma - 1}{2} M_2^2) = (1 + \frac{\gamma - 1}{2} M_1^2)$$

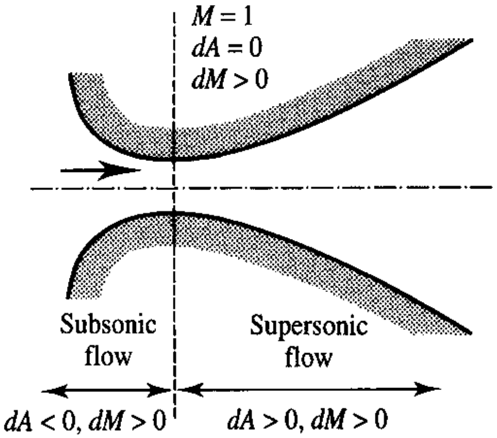
Using (**), we get

$$M_2^2 = \frac{2 + (\gamma - 1) M_1^2}{2\gamma M_1^2 - (\gamma - 1)}$$

Area variation: Convergent –divergent nozzle

Flow in a duct or stream tube with varying cross-section

- Engineering applications:
  - Flow in the nozzle of a rocket engine
  - Flow in a blade passage of a turbo-machine
  - Flow in a supersonic tunnel



- Quasi-1D assumption

- Flow is isentropic except at shock waves

Area variation: Basic relations

- Continuity equation: mass flow rate = const

$$\rho VA = const \Rightarrow \frac{d(\rho VA)}{\rho VA} = 0 \quad \Rightarrow \quad \boxed{\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0}$$

- Energy conservation: total energy =const

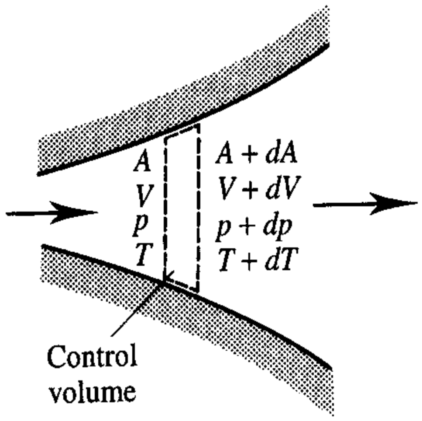
$$c_p T + \frac{V^2}{2} = const \quad \Rightarrow \quad \boxed{c_p dT + VdV = 0}$$

- Perfect gas law: $p=\rho RT$

$$\boxed{\frac{dp}{p} = \frac{d(\rho RT)}{\rho RT} = \frac{d\rho}{\rho} + \frac{dT}{T}}$$

- Isentropic relation: p/ρ^{γ} =const

$$\boxed{\frac{dp}{p} = \frac{\gamma d\rho}{\rho}}$$

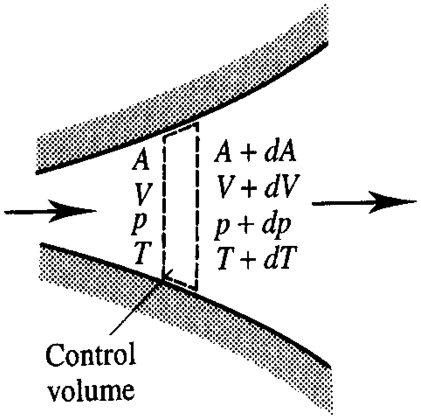


Area variation: Basic relations

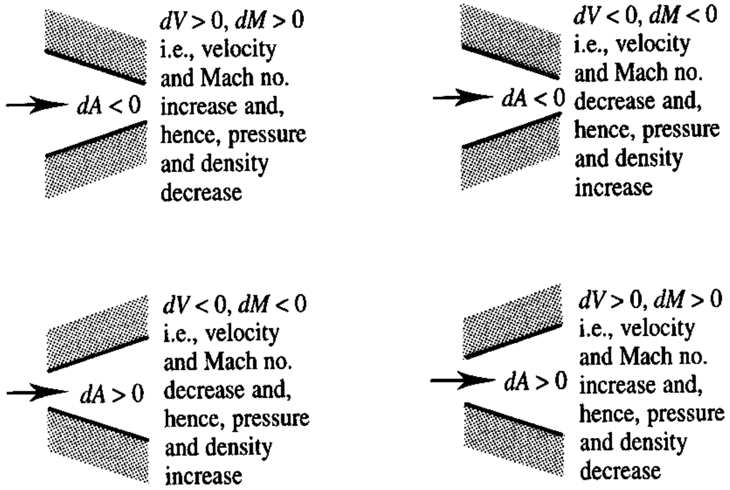
$$\frac{d\rho}{\rho} = -M^2 \frac{dV}{V}$$

$$\frac{dM}{M} = \left[1 + \frac{\gamma - 1}{2} M^2\right] \frac{dV}{V}$$

$$\frac{dA}{A} = \left(M^2 - 1\right) \frac{dV}{V}$$



Area variation: Basic relations

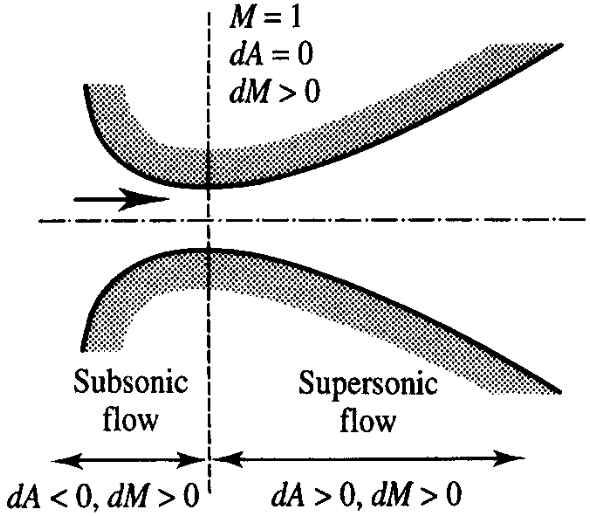


subsonic   supersonic

$$\frac{dA}{A} = \left(M^2 - 1\right)\frac{dV}{V}$$

1. If M<1 or subsonic flow, we have $dA \square dV < 0$
2. If M>1 or supersonic flow, we have $dA \square dV > 0$
3. If M=1 or transonic flow, we have $dA = 0$

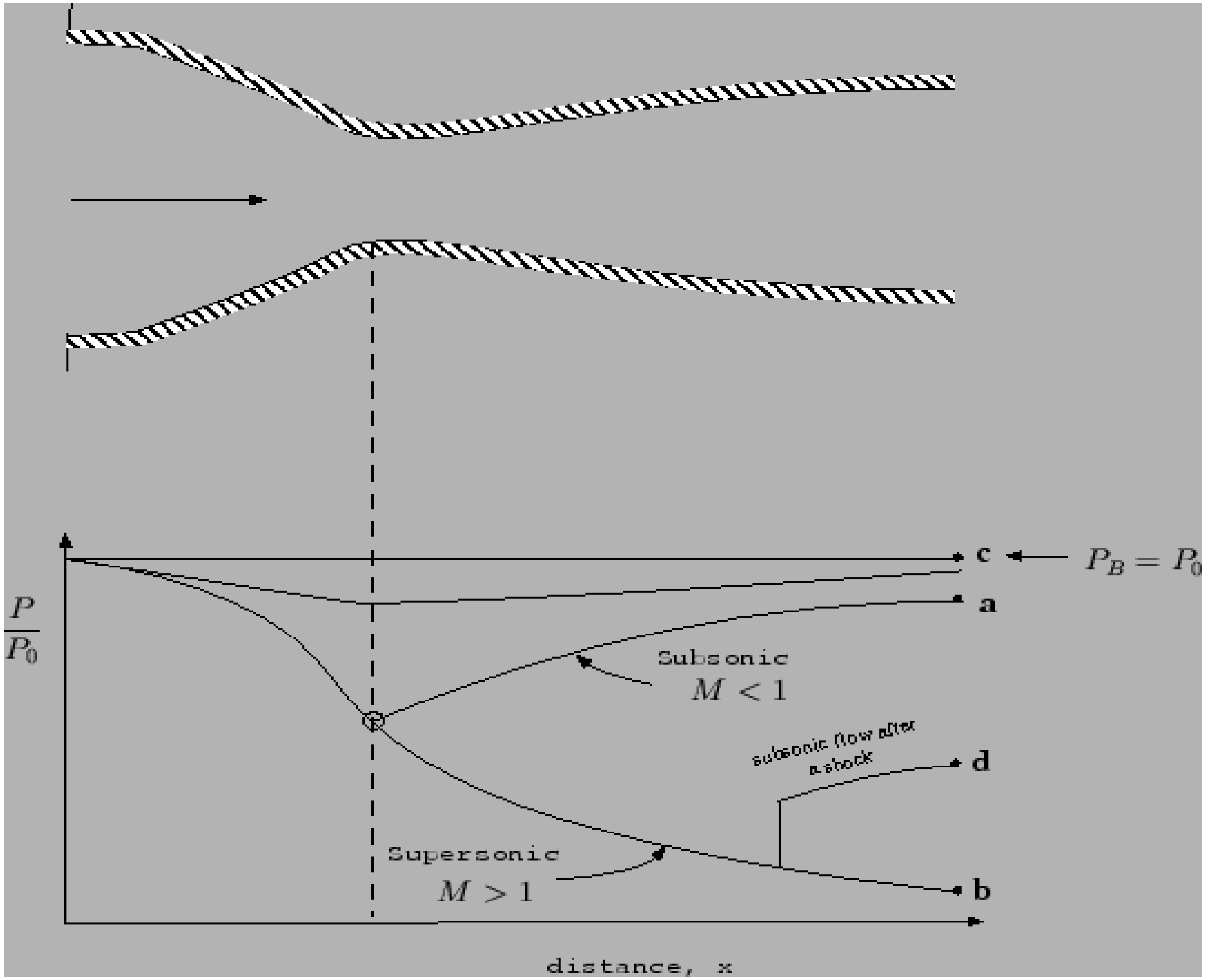
Area variation: Supersonic flow



$$\frac{dA}{A} = \left(M^2 - 1\right)\frac{dV}{V}$$

- Convergent-divergent duct

Converging-diverging nozzle



Adiabatic flow with friction: Fanno line flow

Momentum equation: $\frac{\pi}{4}D^2 \cdot \Delta(p+\rho u^2)+\pi D \Delta x \cdot \tau_w = 0$

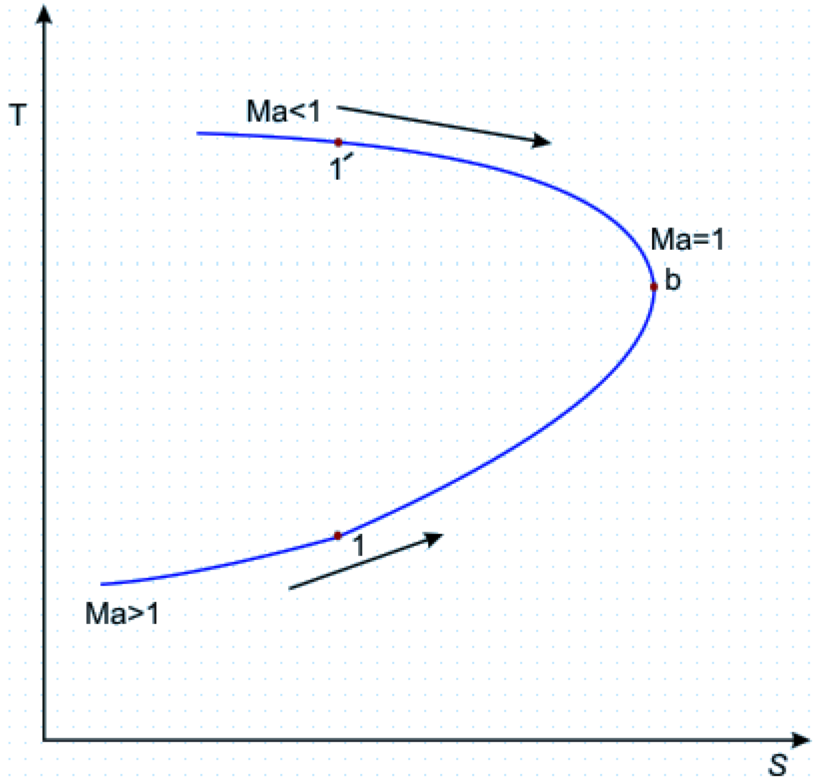
Darcy friction factor: $f = \frac{8\tau_w}{\rho u^2}$

After some manipulations:

$$\left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma}\ln\left(\frac{M^2}{1+\frac{\gamma-1}{2}M^2}\right)\right]_{M_1}^{M_2} = \int_{X_1}^{X_2} \frac{f}{D}dx$$

$$\frac{p_{01}}{p_{02}} = \frac{M_1}{M_2}\left[\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Adiabatic flow with friction: Fanno line flow



Flow with heat exchange: Rayleigh line flow

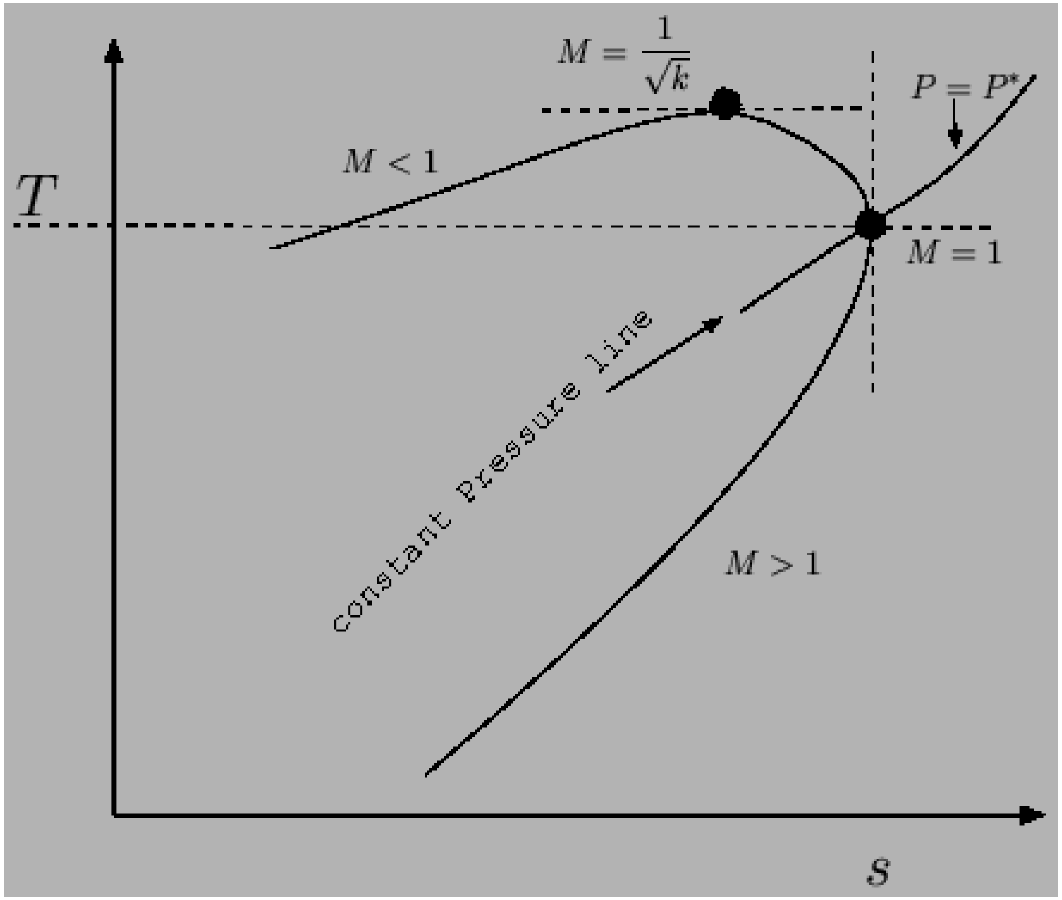
Added heat: $q = c_p (T_{02} - T_{01})$

We get

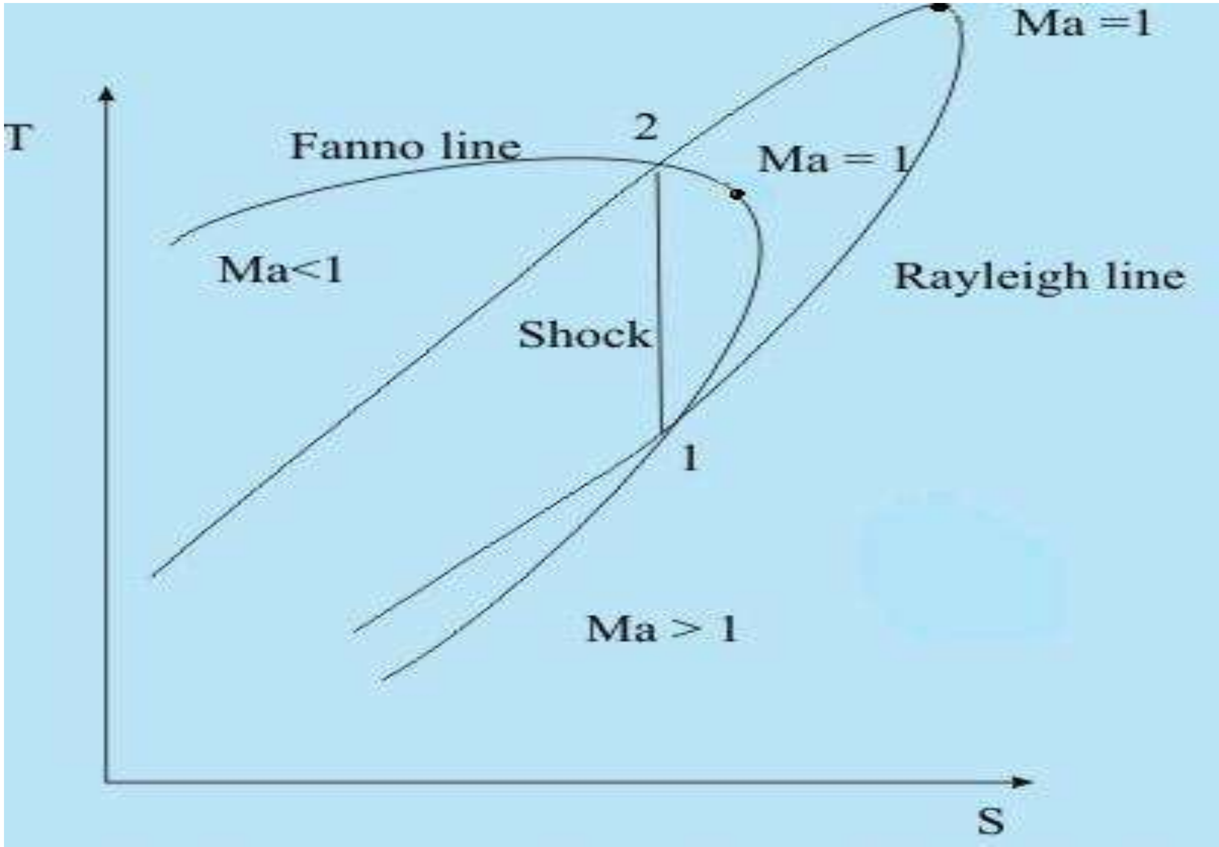
$$s_2 - s_1 = c_p \ln\left[\frac{T_2}{T_1}\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

$$\frac{p_{02}}{p_{01}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}\left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{(\gamma-1)}}$$

Flow with heat exchange: Rayleigh line flow



Location of Shock for M>1



It is easily shown that a possible shock will be located at the intersection between Fanno line and Rayleigh line

The velocity potential equation

Assume zero vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{V} = \mathbf{0}$

Introduce velocity potential: $\mathbf{V} = \nabla \boldsymbol{\Phi}$

Continuity equation: $\nabla \cdot (\rho \mathbf{V}) = \mathbf{0}$

Bernoulli equation: $a^2 d\rho = dp = -\rho V dV = -\frac{1}{2}\rho d\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right)$

Combining the equations, we get

$$\left(1 - \frac{\Phi_x^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right)\Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right)\Phi_{zz}$$
$$-2\frac{\Phi_x\Phi_y}{a^2}\Phi_{xy} - 2\frac{\Phi_x\Phi_z}{a^2}\Phi_{xz} - 2\frac{\Phi_y\Phi_z}{a^2}\Phi_{yz} = 0$$

The velocity potential equation

Speed of sound: $$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right)$$

The small perturbation equation:

$$\mathbf{V} = (U_\infty + u', v', w')$$

$$\mathbf{V}' = \nabla \phi$$

$$\left(1 - M_\infty^2\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

The full compressible flow equations

- Conservation of Mass (Continuity equation)

$$\frac{D\rho}{Dt} + \rho \, div\vec{V} = 0 \qquad \text{or} \qquad \frac{\partial \rho}{\partial t} + div\,\rho\vec{V} = 0$$

- Conservation of Momentum

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda div\vec{V} \right]$$

- Conservation of Energy

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + div\left(k\nabla T\right) + \tau'_{ij}\frac{\partial u_i}{\partial x_j}$$

or, alternatively,

$$\rho c_p \frac{DT}{Dt} = \beta T \frac{Dp}{Dt} + div\left(k\nabla T\right) + \Phi$$

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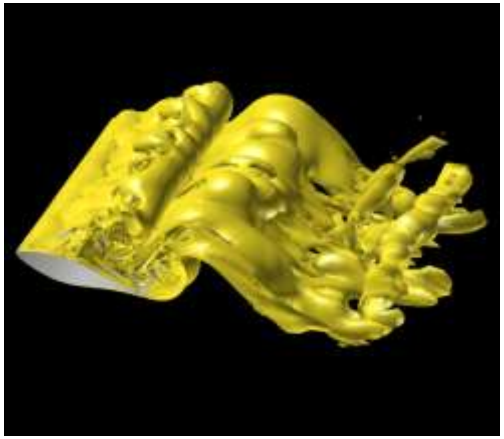
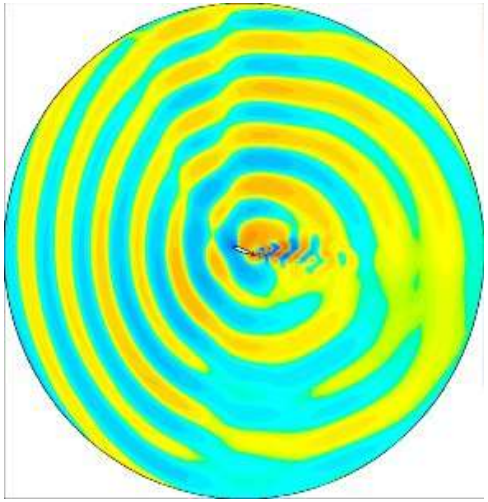
Aero-acoustic (CAA) Modelling

Basic Principles:



- Flow divided into incompressible and acoustic part
- Model integrated with flow solver
- Covers generation as well as emission of noise
- Valid for both laminar and turbulent flows
- Compact high-order schemes introduced



Aeroacoustics: Splitting approach

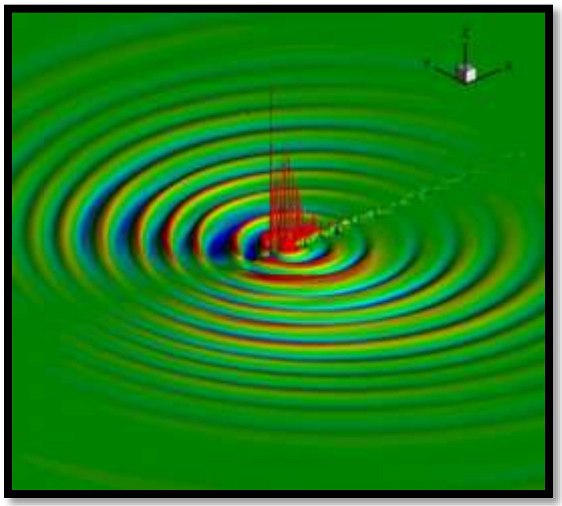
$$\frac{\partial \rho^*}{\partial t} + \frac{\partial f_i}{\partial x_i} = 0$$

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x_j}\left[f_i\left(\bar{U}_j + u_j^*\right) + \rho_0 \bar{U}_i u_j^* + \left(p^* + \frac{2}{3}\rho^* k\right)\delta_{ij}\right] = \frac{\partial}{\partial x_j}\left[\rho^* \nu_t \left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right)\right]$$

$$\frac{\partial p^*}{\partial t} - c^2 \frac{\partial \rho^*}{\partial t} = -\frac{\partial \bar{P}}{\partial t}$$

$$f_i = \rho u_i^* + \rho^* \bar{U}_i \qquad c^2 = \frac{\gamma\left(\bar{P} + p^*\right)}{\rho_0 + \rho^*}$$

Aero-acoustic (CAA) Modelling



Turbulent flow past airfoil